TABLE 1.

LOWERING OF THE FREEZING-POINT IN A MIXTURE OF MANNITE AND POTASSIUM CHLORIDE

CONC. MANNITE	CONC. KCl	MIXTURE	FREEZING-POINT LOWERING		DEVIATION
			Mannite	KCl alone	
0.00493	0.00987	0.04479	0.00927	0.03557	0.00005
0.01071	0.02153	0.09676	0.02012	0.07670	0.00006
0.02183	0.04367	0.19400	0.04061	0.15343	0.00004
0.04067	0.08134	0.35700	0.0757	0.2813	0.0000

So far as the writers have been able to find, this is the first work on mixtures of salts which has been carried out with an accurate temperature measuring system. One set of measurements on mixtures of a salt with a non-electrolyte has been carried out by Osaka [*Zs. physik. Chem.*, **41**, 560 (1902)], but his results show a greater deviation than that illustrated by the preceding table.

We wish to express our indebtedness to the National Academy of Sciences for a grant from the Wolcott Gibbs Fund, which was used for the purchase of the temperature measuring system, to W. P. White for the design of a special potentiometer, and to L. H. Adams and John Johnston of the Geophysical Laboratory of the Carnegie Institution for the loan of the freezing-point apparatus.

CERTAIN GENERAL PROPERTIES OF FUNCTIONS

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Let $g(x)$ be a real, one-valued, bounded, continuous or discontinuous function defined in the interval (a, b). The functional values of $g(x)$ *in the subinterval* (α, β) of (a, b) have a least upper-bound, a greatest lower-bound and a saltus, which we denote respectively by

$$u(g, \alpha\beta),\ l(g, \alpha\beta) \text{ and } s(g, \alpha\beta) = u(g, \alpha\beta) - l(g, \alpha\beta).$$

The upper-bound, the lower-bound[1] and the saltus of $g(x)$ *at the fixed point* x of (a, b) are defined respectively as the (greatest) lower-bound of $u(g, \alpha\beta)$, the (least) upper-bound of $l(g, \alpha\beta)$ and the (greatest) lower-bound of $s(g, \alpha\beta)$ for all possible subintervals (α, β) of (a, b) that contain x as interior point.[2] With the given function $g(x)$, there are thus associated three new functions of x, *the upper-bound function, the lower-bound function* and *the saltus function*, which we denote by

$$u(g, x),\ l(g, x) \text{ and } s(g, x)$$

respectively.[3]

It is seen that

$$s(g, x) \equiv u(g, x) - l(g, x).$$

Let us, for the sake of brevity, write

$$s(g, x) \equiv s'(x),\ s(s', x) \equiv s''(x),\ s(s'', x) \equiv s'''(x),\ \ldots.$$

Sierpiński[4] has proved the

Theorem. $s''(x) \equiv s'''(x) \equiv s^{\text{IV}}(x) \ \ldots.$

The chief aim of the present paper is to communicate a number of companion propositions to Sierpiński's theorem. The new results are based on the definition of other types of saltus, which immediately suggest themselves and arise from the one described above, when certain specified subsets of the range of the independent variable may be neglected.

The first new type arises when *finite* subsets may be neglected. For every subinterval (α, β) of (a, b), there evidently exists a number, which we denote by $u_f(g, \alpha\beta)$, uniquely characterized by the following double property:[5] For every $\epsilon > 0$, on the one hand, the set of points of the interval (α, β) for which $g(x) > u_f(g, \alpha\beta) + \epsilon$ is finite, while on the other hand, there exists an infinite set of points of (α, β) at which $g(x) > u_f(g, \alpha\beta) - \epsilon$. This number $u_f(g, \alpha\beta)$ is the lower-bound of all possible upper-bounds that $g(x)$ may have in the interval (α, β), in case a finite set of points may be neglected. Likewise, there is a number, which we denote by $l_f(g, \alpha\beta)$ characterized by the property that for every $\epsilon > 0$, the set of points of (α, β) where $g(x) < l_f(g, \alpha\beta) - \epsilon$ is finite, whereas, the set of points of (α,β) where $g(x) < l_f(g, \alpha\beta) + \epsilon$ is infinite. $l_f(g, \alpha\beta)$ is the upper-bound of all possible lower-bounds of $g(x)$ in (α, β), when a finite number of points may be neglected. Finally, we denote by $s_f(g, \alpha\beta)$ the lower-bound of the saltus of $g(x)$ in (α, β), in case a finite number of points may be neglected. Evidently

$$s_f(g, \alpha\beta) = u_f(g, \alpha\beta) - l_f(g, \alpha\beta).$$

We shall designate the numbers

$$u_f(g, \alpha\beta),\ l_f(g, \alpha\beta) \text{ and } s_f(g, \alpha\beta)$$

as '*the f-upper-bound,*' '*the f-lower-bound*' and '*the f-saltus*' of $g(x)$ *in the interval* (α, β). As in the case where no point may be neglected, we now define '*the f-upper-bound,*' '*the f-lower-bound,*' and '*the f-saltus*' of $g(x)$ *at the fixed point* x of (a, b), as the lower-bound of $u_f(g, \alpha\beta)$, the upper-bound of $l_f(g, \alpha\beta)$ and the lower-bound of $s_f(g, \alpha\beta)$ for all possible subintervals (α, β) of (a, b) that contain x as interior point. With the given function $g(x)$, we have thus associated '*the f-upper-bound function,*' '*the f-lower-bound function*' and '*the f-saltus function,*' which we denote by

$$u_f(g, x),\ l_f(g, x) \text{ and } s_f(g, x)$$

respectively.

In the second place, the subsets that may be neglected are *denumerable.* As before, there exists a number, which we denote by $u_d(g, \alpha\beta)$, uniquely characterized by the property that, for every $\epsilon > 0$, the set of points of (α, β) where $g(x) > u_d(g, \alpha\beta) + \epsilon$ is denumerable, whereas the set of points where $g(x) > u_d(g, \alpha\beta) - \epsilon$ is non-denumerable; this number $u_d(g, \alpha\beta)$ we call '*the d-upper-bound*' of $g(x)$ in (α, β). Precisely as before we define the related numbers

$$l_d(g, \alpha\beta),\ s_d(g, \alpha\beta),\ u_d(g, x),\ l_d(g, x)$$

and '*the d-saltus function*' $s_d(g, x)$.

In the third place, '*exhaustible*' sets (i.e., sets of first category[6]) may be neglected. We then obtain '*the e-saltus function*' $s_e(g, x)$, together with the related numbers.

Finally, sets of (Lebesgue) *zero measure* may be neglected. We then obtain '*the z-saltus function*' $s_z(g, x)$, together with the related functions.

As in the case where no point may be neglected, we write

$$s_f(g, x) \equiv s_f'(x),\ s_f(s_f', x) \equiv s_f''(x),\ s_f(s_f'', x) \equiv s_f'''(x), \ldots ;$$

and similarly for the d-saltus, the e-saltus and the z-saltus functions.

Having defined the new types of saltus we had in view, we may now state the corresponding analogues of Sierpinski's theorem. The most interesting and least obvious results are the following two theorems.

Theorem. $\qquad s_d'''(x) \equiv s_d^{\mathrm{IV}}(x) \equiv s_d^{\mathrm{V}}(x) \equiv \ldots .$

Theorem. $\qquad s_z'''(x) \equiv s_z^{\mathrm{IV}}(x) \equiv s_d^{\mathrm{V}}(x) \equiv \ldots .$

Moreover, as examples show, $s_d''(x)$ and $s_z''(x)$ may be different from $s_d'''(x)$ and $s_z'''(x)$ respectively.

In the case of the f-saltus, the functions $s_f^{(n)}(x)$ $(n = 1, 2, \ldots)$ *may all be different.*

In the case of the e-saltus, while $s_e''(x)$ may be $\neq s_e'''(x)$, we have

Theorem. $\qquad s_e'''(x) \equiv 0.$

A generalization of our result for the d-saltus is as follows:

Theorem. If $s_{\aleph}'(x)$, $s_{\aleph}''(x)$, $s_{\aleph}'''(x)$ $\ldots$ *represent the successive saltus functions that arise when,* instead of neglecting denumerable sets (i.e., sets of cardinal number $\aleph_0$), *we neglect sets of cardinal number* $\aleph$, *where* $\aleph$ *is any cardinal number* $< c$, the cardinal number of the continuum, *then*

$$s_{\aleph}'''(x) \equiv s_{\aleph}^{\mathrm{IV}}(x) \equiv s_{\aleph}^{\mathrm{V}}(x) \equiv \ldots .$$

Because of our negative result in the case of the f-saltus, we are naturally led to define $s_f^{(\beta)}(x)$ for transfinite β's. Our result will show that it is sufficient to confine ourselves to transfinite numbers belonging to Cantor's second class. If β is not a limiting number, $\beta - 1$ exists, and

we define $s_f^{(\beta)}(x)$ as equal to $s_f(s_f^{(\beta-1)}, x)$. For our purpose, therefore, all we have to do now is to define $s_f^{(\beta)}(x)$, in case β is a limiting number, in terms of the functions $s_f^{(\nu)}(x)$, where $\nu < \beta$. This we do simply by means of the equation

$$s_f^{(\beta)}(x) = \lim_{\nu_n \rightarrow \beta} s_f^{(\nu_n)}(x),$$

where $\{\nu_n\}$ is a sequence of numbers less than β. It is seen that this limit always exists and is independent of the particularly chosen sequence $\{\nu_n\}$.

Our positive result for the case of the f-saltus may now be stated as follows:

Theorem. There exists a number β of the first or the second class, such that

$$s_f^{(\beta)}(x) \equiv s_f^{(\beta+1)}(x).$$

Furthermore, it is shown by means of an example, that *if β is a given number of the first or the second class, then the functions $s_f^{(\nu)}(x)$ $(1 \leqq \nu \leqq \beta)$ may all be different, whereas $s_f^{(\beta)}(x) \equiv s_f^{(\beta+1)}(x)$.*

The following interesting connection exists between the d-saltus and the f-saltus.

Theorem. For every function $g(x)$ for which $s_f^{(\beta)}(x) \equiv s_f^{(\beta+1)}(x)$ —— and according to the preceding theorem there always exists such a β of the first or the second class ——, *we have*

$$s_d(s_f'', x) \equiv s_f^{(\beta)}(x).$$

The results may be easily extended to the case of many-valued, bounded, or unbounded functions of several variables, or of infinitely many variables; and by means of simple postulate systems, to more abstract situations.

The above is essentially the introduction of a paper, which is to be offered to the *Annals of Mathematics*.

[1] Throughout the paper, we use the expressions, 'the upper-bound' and 'the lower-bound,' in the sense of 'the least upper-bound' and 'the greatest lower-bound,' respectively.

[2] Of course, it will be understood, that in case $x = a, b$—and only then—we permit α, β to coincide with x. This remark applies also to similar situations below.

[3] The functions $u(g, x)$ and $l(g, x)$ are often, though not quite unobjectionably, called the 'maximum' and the 'minimum' functions belonging to $g(x)$. Cf., for example, Hobson, *The Theory of Functions of a Real Variable* (1907), art. 180.

[4] *Bull. Acad. Sci., Cracovie* (1910), 633–634.

[5] Cf. Baire, *Acta Mathematica*, **30**, 21 and 22 (1906).

[6] Cf. Denjoy, *J. math. Paris*, Ser. 7, **1**, 122–125 (1915).